Divisibility of 2n choose n by a prime.

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A recent question and answer sent to Quandaries and Queries was

"Are there infinitely many n such that 105 "divides" 2n choose n?

Your question is an unsolved problem.

Erdos, Graham, Rusza and Straus (Math. of Comp., 29(1975),pp 83-92) show that for any two primes p and q there exist infinitely many integers n for which (C(2n,n),pq) = 1. They remark that nothing is known for three primes and, in particular, they ask whether there are infinitely many n for which (C(2n,n),105) = 1 and this is your problem. As far as I know this is still unsettled.

If we want to look at prime factors of n! there is a nice way to find the power to which a prime p occurs dating back to Legendre. Legendre observed that if $\alpha_p(n)$ is the power to which p divides n! then

$\alpha_p(n) = \left[\frac{n}{p^1}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \dots$ where the square brackets [] denote the greatest integer function. For example, the largest power of 3 in 204! is

$$\alpha_3(204) = \left[\frac{204}{3^1}\right] + \left[\frac{204}{3^2}\right] + \left[\frac{204}{3^3}\right] + \left[\frac{204}{3^4}\right] + \dots = 68 + 22 + 7 + 2 = 99.$$

This is because 3 divides every 3rd number in the sequence 1, 2, 3, ... , 204; 3^2 divides every 9th number in the sequence 1, 2, 3 , ... , 204; and 3^3 divides every 27th number in the sequence 1, 2, 3, ... , 204; and 3^4 divides every 81st number in the sequence 1, 2, 3, ... , 204.

Similarly if you wanted to find out how many zeroes appear at the end of 204! what you really need to find out is how often 5 divides 204! That's

$$\alpha_5(204) = \left[\frac{204}{5^1}\right] + \left[\frac{204}{5^2}\right] + \left[\frac{204}{5^3}\right] + \dots = 40 + 8 + 1 = 49.$$

Note that if we write 204 in base 3 (see Converting to other bases in Quandaries & Queries), i.e. 204 = 21120_3 the sum of the digits in base 3 is 6. If we write 204 in base 5 we get 204 = 134_5 the sum of the digits is 8. Observe that (204 - 6)/(3 - 1) = 99. Observe also that

(204 - 8)/(5 - 1) = 49. Is this an accident, getting 99 and 49 again this way? It isn't. The proof is a little messy but not too hard, however we won't go through it here. Let's have a peek at what goes on though.

Let me write Sp(n) for the sum of the digits of n in base p. We can in general show that the power to which p divides n! can be expressed as $\alpha_p(n) = \frac{n - S_p(n)}{p-1}$. The interested reader might then want to see how often a prime p divides $\binom{2n}{n}$, as was asked in the Quandaries and Queries question, call it $\beta_p(n)$; we find, looking at the factorials in the numerator and denominator of $\binom{2n}{n}$ that

$$\beta_p(n) = \alpha_p(2n) - 2\alpha_p(n) = \frac{2n - S_p(2n)}{p-1} - 2\frac{n - S_p(n)}{p-1} = \frac{2S_p(n) - S_p(2n)}{p-1}.$$

For example, 204 = 21120_3 and 102 = 10210_3 . Thus $\beta_3(204) = \frac{2(4)-6}{3-1} = 1$. Similarly 204 = 134_5, 102 = 42_5 thus $\beta_5(204) = \frac{2(6)-8}{5-1} = 1$. That is both 3 and 5 divide 204 choose 102 to the first power only.
