

## Math Workshop October 2010

### Fractions and Repeating Decimals

This evening we will investigate the patterns that arise when converting fractions to decimals. As an example of what we will be looking at,

1. Write  $3/7$  as a repeating decimal by dividing 7 into 3.0000... . How many digits are in the repetend (i.e. in the repeating block of digits)? Add the number formed by the first half of the repetend to the number formed by the second half.

We start by reviewing sequences.

#### Adding arithmetic and geometric sequences.

There are two types of sequences that appear so often that we give them names. To get any term after the first in an **arithmetic sequence**, you add a fixed constant to the previous term; for example,

7, 10, 13, 16, 19, 22,

where the difference between consecutive terms is  $d = 3$ . For a **geometric sequence** you multiply the previous term by a fixed constant; for example,

7, 21, 63, 189, 567.

Here the ratio of consecutive terms (one term divided by the previous term) is  $r = 3$ .

There are simple tricks for adding the terms of these sequences. To easily compute the arithmetic sum  $S = 7 + 10 + 13 + 16 + 19 + 22$ , write the sum forwards and backwards, then add vertically:

$$\begin{array}{r} S = 7 + 10 + 13 + 16 + 19 + 22 \\ S = 22 + 19 + 16 + 13 + 10 + 7 \\ \hline 2S = 29 + 29 + 29 + 29 + 29 + 29 = 6 \cdot 29; \end{array}$$

Since  $2S = 6 \cdot 29$ , we deduce that  $S = 3 \cdot 29 = \underline{87}$ , which can easily be verified by adding the given six numbers.

And here is the trick for geometric sums. To compute a geometric sum such as  $S = 7 + 21 + 63 + 189 + 567$ , subtract  $S$  from  $3S$  (where the number we multiply  $S$  by is the common ratio of any two consecutive terms,  $3 = 21/7 = 63/21 = 189/63 = \dots$ ):

$$\begin{array}{r} 3S = 21 + 63 + 189 + 567 + 1701 \\ - S = - (7 + 21 + 63 + 189 + 567) \\ \hline 2S = -7 + 1701; \end{array}$$

thus,  $2S = 1694$ , whence  $S = \underline{847}$ , which again is easily checked directly. Here are four practice problems. Although there are formulas that give you the sum from the initial and final elements and the size of the jump, *understanding the trick is much more important than mastering the formulas.*

2. Compute the following four sums.

- a. Sum the odd numbers between 10 and 1000.

- b. Sum the powers of 4 from 4 to 1024.

- c.  $\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \frac{7}{100000}$ . (Suggestion. Use  $10S$  instead of  $S/10$ .)

- d.  $\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \frac{7}{100000} + \dots$  (where the three dots mean that you should continue the sum to infinity).

**Fractions and repeating decimals.**

Parts (c) and (d) of Exercise 2 suggest how geometric sums might be related to rational numbers, which is the main topic of today's workshop. We define a **rational number** to be any real number that can be written as a fraction; a **fraction** is any number that can be written in the form  $\frac{a}{b}$ , where  $a$  and  $b$  are integers and  $b \neq 0$ . Here are some examples:

$$\frac{2}{5}; \frac{5}{2}; .8 = \frac{8}{10}; 4\frac{1}{3} = \frac{13}{3}; -18 = -\frac{18}{1}; -\frac{4}{3} = \frac{-4}{3} = \frac{4}{-3}; 0 = \frac{0}{1}; 11.37 = \frac{1137}{100}.$$

Every integer is a rational number since the integer  $n$  can be written as  $\frac{n}{1}$ . It is obvious that  $\sqrt{81}$  is rational because it equals  $9 = \frac{9}{1}$ . On the other hand, it is entirely unobvious that  $\sqrt{2}$  is *not* rational:  $\sqrt{2}$  *cannot* be written as a fraction, a discovery that came as quite a surprise to the ancient Greeks. The proof of the irrationality of  $\sqrt{2}$  is not hard, but we will leave it for another time. Tonight we are interested in exploring another unobvious fact: the rationals are precisely those numbers that can be written as repeating decimals. A **repeating decimal** is a number written in decimal notation whose digits to the right of the decimal point eventually consist of endless repetitions of a block of numbers called the *repetend*. The number of digits in the smallest repeating block is called the **period** of the repeating decimal. For example,  $\frac{1}{3} = .3333 \dots$  has period 1, and  $\frac{2779}{110} = 25.2636363\dots$  has period 2. We use a bar over the repeating part of the decimal as shorthand; thus

$$\frac{1}{3} \text{ is written as } 0.\overline{3} \text{ (or just } \overline{.3}), \text{ while } \frac{2779}{110} \text{ becomes } 25.\overline{263}.$$

$\frac{1}{4} = .250000000 \dots$  is also a repeating decimal, which we usually write as  $.25$  (or  $0.25$ ). (Numbers where the repeating part is a zero are called "terminating decimals" by schoolteachers, but we have no need here for more terminology.)

We can write any fraction as a repeating decimal by using long division. An example will make this clear. Let's write  $345/22$  as a decimal:

$$\begin{array}{r} 15.681 \\ 22 \overline{) 345.000} \\ \underline{22} \phantom{00} \\ 125 \phantom{0} \\ \underline{110} \phantom{0} \\ 150 \phantom{0} \\ \underline{132} \phantom{0} \\ 180 \phantom{0} \\ \underline{176} \phantom{0} \\ 40 \phantom{0} \\ \underline{22} \phantom{0} \\ 18 \end{array}$$

We stopped when 18 repeated as the remainder — we would just bring down the zero again, find that 22 goes into 180 eight times with a remainder of 4, bring down the zero and get 22 goes into 40 one time with a remainder of 18, and continue getting remainders of 18 and 4 forever. We conclude that

$$\frac{345}{22} = 15.6818181\dots = 15.\overline{681}.$$

We knew that eventually the remainders would start to repeat because there are only 22 possible remainders when dividing by 22, namely the integers from 0 up to 21. It just happens that the period is 2 and not something longer. In fact it is easy to see that  $a/b$  will have period 2 if  $b = 2^m \cdot 5^n \cdot 11$  and  $a$  is any integer that is not divisible by 11; furthermore, the sum of the digits of the repeating part will be 9.

3. Verify this for yourself — without a calculator — by showing that  $x = 7/440$  has period 2, but save yourself time and effort by first multiplying  $x$  by  $10^3$  (because  $440 = 2^3 \cdot 5 \cdot 11$ , so 3 is the highest power to which 2 or 5 has been raised), then divide top and bottom of  $10^3x$  by 40 to reduce the fraction to an integer over 11, do the division, then solve for  $x$ .

In the other direction we recognize that  $15.6818181\dots$  is just 15.6 plus a geometric sequence!  $15.6 + .0818181\dots = 15.6 + .081 + .00081 + .000081 + \dots$ , where each term is  $1/100$  times the previous term:

4. Subtract  $x = 15.68181\dots$  from  $100x$ , then solve for  $x$ . Reduce  $x$  to a fraction in lowest terms. (Remember that a fraction is the quotient of integers; you might have to multiply top and bottom by a power of 10 to get it in the proper form.)

For a fraction  $a/b$  in lowest terms, the longest its period could possibly be is  $b - 1$  (because there are  $b - 1$  nonzero possible remainders). To simplify matters, we will restrict ourselves to the case where  $b$  is a prime number, and  $a < b$ . When the period of  $1/b$  is  $b - 1$ , we will call  $b$  a *full-period prime*. The smallest full-period prime is 7.

5. a. Write  $1/7$  as a repeating decimal.  
b. Use part (a) to guess what the decimal expansion of  $2/7$  must be — in particular, you should note where the remainder of 2 shows up in your division of 7 into 1.

Observe the evident patterns in Figure 1. Each fraction  $a/7$  (in the box on the left) has the same cycle of digits in its decimal expansion, but with different starting points. Thus, to compute  $a/7$ , don't think of multiplying the decimal for  $1/7$  by  $a$ ; rather, apply a cyclic permutation to the digits in the repeating block 142857. In this context it is natural to arrange the digits around a circle as shown in the figure. An arrow in the middle reminds us which way to order the digits, and each fraction  $a/7$  is placed outside the circle beside the starting point for its decimal. Thus, looking at the circle diagram, we can read off the decimal expansion of  $3/7$  as follows. The first digit, 4, is inside the circle next to the label  $3/7$ . Starting at that point, proceed around the circle reading off digits 4285714285714285...

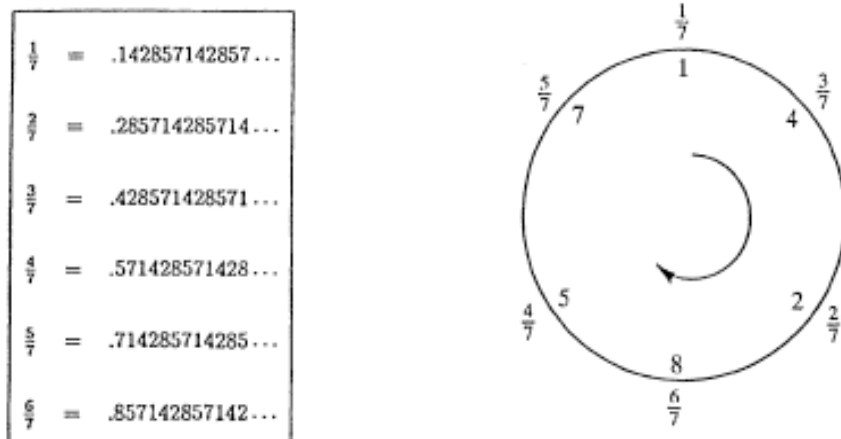


Figure 1. Left: decimal fractions  $m/7$ . Right: circle diagram for 7.

Did you notice that the diametrically opposed digits inside the circle add to 9? Or that the diametrically opposed fractions outside the circle add to 1? Are these just coincidences, or do these patterns show up for denominators other than 7? It turns out that these questions are easily answered for full-period primes, so we will focus on them.

It is not at all clear which primes have a full period. On the last page we include a list of the values of  $1/b$  for integers  $b \leq 50$ . The full-period primes on that list are 7, 17, 19, 23, 29, and 47. (There are many larger full-period primes — 59, 61, 97, 109, ... — but nobody knows whether or not the list is infinite!) Here is one simple method for determining the period corresponding to a prime number:

**Theorem 1.** *For any prime number  $p$  larger than 5, the number of ones in the first element of the sequence  $11, 111, 1111, \dots$  that is divisible by  $p$  equals the period of  $1/p$ .*

6. Test the theorem for the primes 7, 11, and 13 (with periods 6, 2, and 6, respectively). You can easily prove it by applying the method used in problem 4 above to convert  $1/p$  to its decimal form: If  $k$  is the period, then  $(10^k - 1)/p$  will be an integer, call it  $n$ ; that is,  $9 \cdot 111 \dots 1/p = n$ . Since  $p$  is a prime larger than 5, it cannot divide evenly into 9, so it must divide the factor  $111 \dots 1$ , as claimed. Go through this proof using 7 and 11 to better understand the argument—what does  $n$  equal when  $p = 7$ ?  $p = 11$ ?

Consider the circle diagram for  $b = 17$ .



Figure 2. Circle diagram for 17.

7. Your calculator won't help much in this exercise because the period of  $1/17$  is 16, but the figure has all the information you need. You should check (by long division) that the decimal expansion of  $1/17$  begins .05882...; that the first remainder after the decimal point is 10 (that is 17 goes into 1.0 zero times with a remainder of 10); that 17 goes into 100 five times with a remainder of 15 (Look where the 5 and 15 appear on the circle in Figure 2.); that 17 goes into 150 eight times with remainder 14. Furthermore, check that  $15/17$  begins .88235... . It should be clear to you that the decimal expansion of  $15/17$  would contain the same digits in the same order as  $1/17$ , but it begins where the long division of 17 into 1 has a remainder of 15 (so that the first step is to divide 17 into 150).

It should be clear from these examples that when  $b$  is a full-period prime, the  $b - 1$  distinct fractions  $a/b$  with  $1 \leq a \leq b - 1$  have repetends that are cyclic permutations of a common sequence of digits  $d_1, d_2, d_3, \dots, d_{b-1}$ .

8. Explain why  $b$  must be prime when  $1/b$  has period  $b - 1$ . (Hint. If, to the contrary,  $b = pq$  had period  $b - 1$ , what would you know about the period of  $p/b = 1/q$ ?)

$d_1, d_2, d_3, \dots, d_{2k}$ , then  $d_1 + d_{k+1} = d_2 + d_{k+2} = \dots d_k + d_{2k} = 9$ .

- c.  $p$  divides  $(10^{2k} - 1)/9 = \underbrace{111\dots1}_{2k}$ , but fails to divide any shorter string of 1s. In particular,  $p$  does not divide  $\underbrace{111\dots1}_k$ .

e. The decimal expansion of the sum  $10^k/p + 1/p$  may be represented this way:

[illegible]

- You may have noticed that in the proof of Theorem 2 we made little use of the condition that  $p$  be a full period prime. In fact, we used only the fact that the period is even. An 1836 theorem of the French mathematician E. Midy states that the property of  $1/p$  in Theorem 2 holds for the fraction  $a/p$ , where  $p$  is an arbitrary prime larger than 5 and  $a$  an integer that is not a multiple of  $p$ , *as long as the period of  $p$  is even*. Observe in the table on the last page that the prime numbers from 31 to 43 all have odd periods so that Midy's theorem does not apply to them. It does apply to the other primes on the list. When  $p$  is not a full period prime, the period of  $1/p$  is a divisor of  $p - 1$ , and for each  $a$  that is not a multiple of  $p$ ,  $a/p$  has the same period as  $1/p$ .

11. Draw the circle diagrams for  $a/13$ ,  $a < 13$ . Here the circle diagrams come with six fractions each; one circle can start with  $1/13$ , the other with  $2/13$ .

Consider the long-division method you used with Exercise 11. At each step we bring down a zero, divide by 13, and subtract to obtain the new remainder. But "bring down a zero" really means multiply by 10. That is, if  $r$  is the remainder at a particular step in the division, the next remainder will be  $10r \pmod{13}$ . So when expanding  $1/13$ , starting with the first remainder, 10, the sequence of remainders is  $\mathbf{10} \pmod{13}$ ,  $\mathbf{10^2 = 9} \pmod{13}$ ,  $\mathbf{10^2 = 9} \pmod{13}$ ,  $\mathbf{10^3 = 12} \pmod{13}$ ,  $\mathbf{10^4 = 3} \pmod{13}$ ,  $\mathbf{10^5 = 4} \pmod{13}$ . (Please note how these numbers correspond to the fractions 10/13, 9/13, 12/13, 3/13, and 4/13 about the first circle you made in Exercise 11.) The sequence repeats when we reach a power of 10 that returns to the starting point:  $\mathbf{10^6 = 1} \pmod{13}$ ; that is, 13 divides evenly into  $10^6 - 1$ . Since  $10^6 - 1 = (10^3 + 1)(10^3 - 1)$ , and 13 does not divide  $10^3 - 1$  (because if it did, we would have found the period of  $1/13$  to be shorter than 6), we

deduce that 13 must divide  $10^3 + 1$ . (Of course, it's easy to check that  $1001 = 7 \cdot 11 \cdot 13$ .)

This approach gives us a second way to recognize full-period primes:

**Theorem 3.**  $1/p$  has period  $p - 1$  if and only if every positive integer less than  $p$  is congruent to  $10^j \pmod{p}$  for some  $j < p$ .

Table of the decimal expansions of  $1/x$  for  $x \leq 50$ .

$x$	$1/x$	$x$	$1/x$
1	$\overline{9}$	26	$\overline{.0384615}$
2	$\overline{.49}$	27	$\overline{.037}$
3	$\overline{3}$	28	$\overline{.03571428}$
4	$\overline{.249}$	29	$\overline{.0344827586206896551724137931}$
5	$\overline{.19}$	30	$\overline{.03}$
6	$\overline{.16}$	31	$\overline{.032258064516129}$
7	$\overline{.142857}$	32	$\overline{.031249}$
8	$\overline{.1249}$	33	$\overline{.03}$
9	$\overline{.1}$	34	$\overline{.02941176470588235}$
10	$\overline{.09}$	35	$\overline{.0285714}$
11	$\overline{.09}$	36	$\overline{.027}$
12	$\overline{.083}$	37	$\overline{.027}$
13	$\overline{.076923}$	38	$\overline{.0263157894736842105}$
14	$\overline{.0714285}$	39	$\overline{.025641}$
15	$\overline{.06}$	40	$\overline{.0249}$
16	$\overline{.06249}$	41	$\overline{.02439}$
17	$\overline{.0588235294117647}$	42	$\overline{.0238095}$
18	$\overline{.05}$	43	$\overline{.023255813953488372093}$
19	$\overline{.052631578947368421}$	44	$\overline{.0227}$
20	$\overline{.049}$	45	$\overline{.02}$
21	$\overline{.047619}$	46	$\overline{.02173913043478260869565}$
22	$\overline{.045}$	47	$\overline{.0212765957446808510638297872340425531914893617}$
23	$\overline{.0434782608695652173913}$	48	$\overline{.02083}$
24	$\overline{.0416}$	49	$\overline{.020408163265306122448979591836734693877551}$
25	$\overline{.039}$	50	$\overline{.019}$