Two Parabolas

Kind

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The short answer: *Yes*, it is always possible to find the common tangents of two parabolas — the number of common tangents is always either 1 or 3. But even using a computer algebra system, the result would be so intimidating that it would be of no value. A more reasonable approach would be to start with one of the parabolas in standard position. When working with tangents, it is usually easier to use line coordinates [u,v,w], which stand for the line

$$ux + vy + w = 0, \quad \text{or} \quad y = -\frac{u}{v}x - \frac{w}{v}.$$

The equation of a parabola using line coordinates is then

$$u^2 + 2auv + bv^2 + 2cu + 2vd = 0,$$

which is satisfied by the "line at infinity" $u = v = 0$. (*Explanation*: The equation represents what is called the *dual* of the parabola: a line $[u, v, w]$ is tangent to the curve if and only if its coordinates satisfy the line equation. A parabola is defined to be a conic that is tangent to the line at infinity, which means that [0,0,1] satisfies its line equation. You can pass between the equation of a conic and its dual using the corresponding matrix and its inverse, as explained in projective geometry textbooks.) To find the tangents common to the general parabola and the convenient parabola,

$$v = u^2,$$

just plug u^2 in for v in the equation for the general parabola to get $u^2 + 2au^3 + bu^4 + 2cu + 2du^2 = 0$ which, after dividing through by u, is the cubic equation

$$bu^3 + 2au^2 + (2d + 1)u + 2c = 0.$$

(To find the resulting tangent lines, one must solve the cubic, which is guaranteed one real solution. If s is a solution, then the corresponding common tangent will be $[s, s^2, 1]$.)

On the other hand, it sounds from your question that what you really want is a pair of parabolas described in terms of focus and directrix. So given foci A and B and corresponding directrices a and b, one gets any common tangent as a line in which the reflection simultaneously takes A to a point A' on a, and B to a point B' on b. In terms of paper folding, the Beloch Fold is a single fold that places A on a and B on b. A recent paper by Thomas C. Hull ("Solving Cubics with Creases: The Work of Beloch and Lill"; *American Mathematical Monthly*, **118**:4, April 2011, 307-315) describes how to make the fold with an actual piece of paper. For the algebraic version of constructing $\sqrt[3]{r}$,

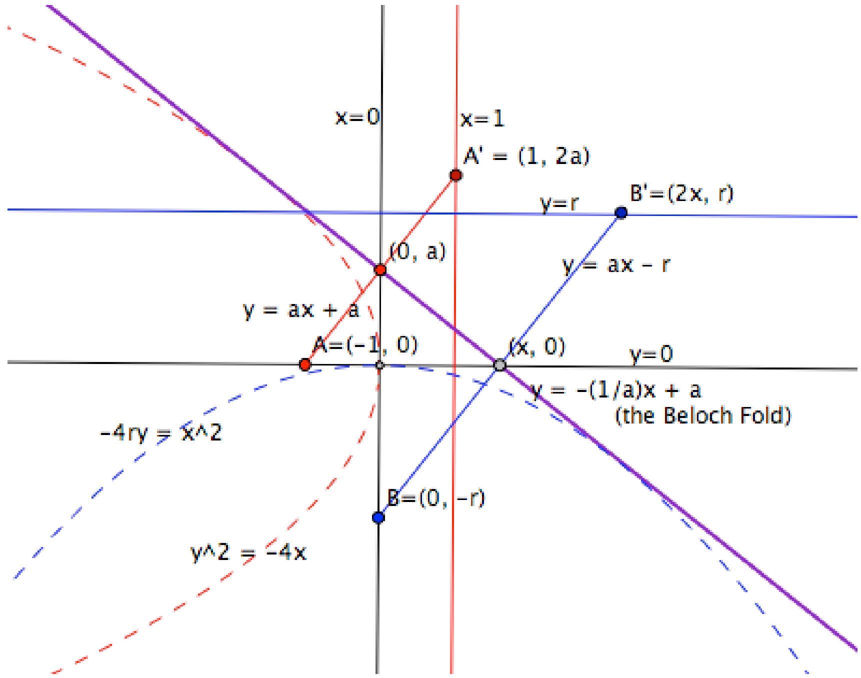

Figure 1: Folding along the purple line places $A : (-1, 0)$ on $x = 1$ and $B : (0, -r)$ on $y = r$.

start with focus $A = (-1, 0)$, directrix $x = 1$, and focus $B = (0, -r)$, directrix $y = r$. (We won't need them, but the corresponding conics are $y^2 = -4x$ and $-4ry = x^2$.) To find the equation of the fold line, start with an arbitrary point $(1, 2a)$ on $x = 1$. The fold line $y = -\frac{1}{a}x + a$ is the perpendicular bisector of the segment joining $(-1, 0)$ with $(1, 2a)$. The goal is to find the value of the variable a so that the fold line is also the perpendicular bisector of the segment joining $(0, -r)$ to a point of the target line $y = r$. This joining line has the equation $y = ax - r$ (namely, the line through $(0, -r)$ that is perpendicular to the fold line); it must intersect the fold line on the line $y = 0$, halfway between $y = -r$ and $y = r$. In terms of the variable x these intersecting lines are

$$x = -ay + a^2 \quad \text{and} \quad x = \frac{y}{a} + \frac{r}{a},$$

whose x values must be equal at the intersection point:

$$-ay + a^2 = \frac{y}{a} + \frac{r}{a},$$

which reduces to

$$y = \frac{a^3 - r}{a^2 + 1}.$$

This variable y can be zero if and only if $a^3 = r$; in other words, the constructed point $(0, a)$ produces the cube root of the arbitrary real number r.

If you start with the point $A = (0, 2r)$ and its target point $(2a, 0)$ on the line $y = 0$, together with $B = (0, s)$ and its target line $x = -2b$ (where $b \neq 0$), the folding line will be $y = \frac{a}{r}x + \left(r - \frac{a^2}{r}\right)$ with a any zero of $x^3 + bx^2 + (rs - r^2)x + br^2 = 0$. Thus, as long as $b \neq 0$, to construct the zeros of the cubic equation

$$x^3 + bx^2 + cx + d = 0,$$

simply set $r = \sqrt{\frac{d}{b}}$ and $s = \sqrt{\frac{d}{b}}\left(c + \frac{d}{b}\right)$.