Chalking out Some Geometry from a Bit of Trigonometry

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Dozens of identities that we learn were discovered overseas a thousand years ago. A couple that we don't learn[1] were discovered ten years ago right here in Canada:

$$\sin^{-1}x + \sin^{-1}y = 2\sin^{-1}v\,,$$
$$\cos^{-1}x + \cos^{-1}y = 2\cos^{-1}v\,, \qquad \textbf{(1)}$$
$$\text{where} \quad 2v = \sqrt{(1+x)(1+y)} - \sqrt{(1-x)(1-y)}\ .$$

They work well for Cartesian geometry[2] and are naturally extendable to Euclidean one.

Proposition 1. Diameter $PQ = 2c$ of semicircle (Diagram 1) has perpendiculars of lengths a, μ, and b. Show that $AM = MB$ **iff** $2\mu = \sqrt{(c+a)(c+b)} \pm \sqrt{(c-a)(c-b)}$ **(2)** .

One way to prove Proposition 1 is to use identities (1) along with arc midpoint computation.

The following statements continue geometric interpretations of (2) and can be deduced similarly. Consider these proofs as exercises.

Proposition 2. A circle (Diagram 2) of diameter c has chords $AC = a$, $MC = \mu$, and $BC = b$. Prove that $\angle ACM = \angle MCB$ **iff** (2).

Proposition 3. Trapezoid (Diagram 3) has bases a, b, and circumdiameter c. Show that the length of its diagonal, μ, satisfies (2).

Proposition 4. Prove that on a globe[3] of radius c (Diagram 4), the parallel of radius μ is equidistant from the parallels of radii a and b **iff** (2).

Diagram 1

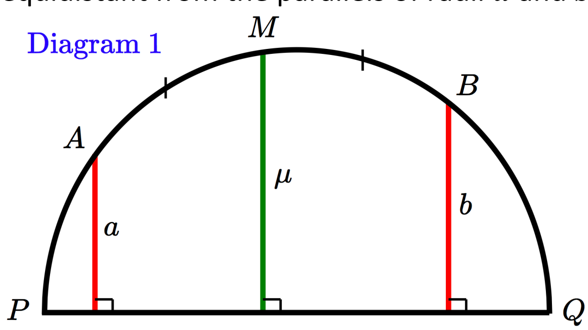


Diagram 2

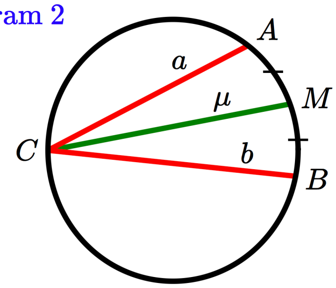


Diagram 3

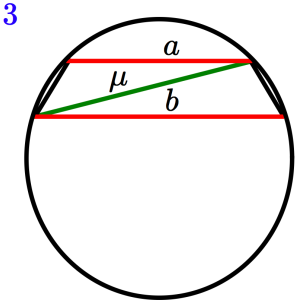
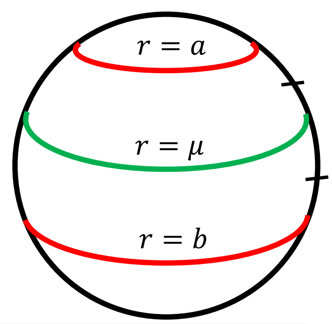


Diagram 4



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[1] http://mathcentral.uregina.ca/RR/database/RR.09.18/akulov3.pdf
[2] http://mathcentral.uregina.ca/RR/database/RR.09.10/akulov2.html
[3] http://mathcentral.uregina.ca/RR/database/RR.09.14/akulov2.html