1 Geometrical Algebra in 2 dimensions

Geometrical Algebra (see http://www.mrao.cam.ac.uk/∼clifford/) use vectors $\vec{e}_1$, $\vec{e}_2$ and build with it and with the commutator and the anticommutator. The scalar "product" is then

1.1 $$\tfrac{1}{2}\{\vec{e_1};\vec{e_2}\} = 0$$

while in two dimension the commutator is a unique element which "represent" the surface object η:

1.2 $$\eta \overset{\text{def}}{=} \tfrac{1}{2}\,[\vec{e_1};\vec{e_2}]$$

The algebra build with $\vec{e_1}$, $\vec{e_2}$ and η is very simple:

1.3

.	$\vec{e_1}$	$\vec{e_2}$	η
$\vec{e_1}$	1	η	$\vec{e_2}$
$\vec{e_2}$	$-\eta$	1	$-\vec{e_1}$
η	$-\vec{e_2}$	$\vec{e_1}$	-1

The algebra says that η anticommute with all the vector $\eta\,\vec{a} = -\vec{a}\,\eta$. A very simple properties that will be used here is related to the rotation of $\vec{a}$ by an angle θ in the trigonometric sens ⤹ :

1.4 $$\vec{a}' = e^{-\eta\theta/2}\vec{a}e^{\eta\theta/2} = e^{-\eta\theta}\vec{a} = \vec{a}e^{\eta\theta} \quad \text{so} \quad \eta\,\vec{a} \text{ is } \perp \text{ to } \vec{a}$$

If we have 2 vectors $\vec{a}$ and $\vec{b}$ and if $S_\triangle$ is the surface of the triangle based on $\vec{a}$ and $\vec{b}$, then

1.5 $$[\vec{a},\vec{b}] = 4\,\eta\,S_\triangle$$

$S_\triangle > 0$ if the $\triangle$ is oriented ⤹ or < 0 if $\triangle$ ⤸.
Using Geometrical Algebra unable us to avoid trigonometry, coordinates, analytical geometry, ... we can just use very simple, non-commutative, vector algebra. In the plane everything would "ressemble" to complex number manipulation and indeed there is a isomorphism between geometrical algebra and the complex plane. But the power of GA is greater and could be extend to any dimension.
For example we have some nice properties connected to the triangle building by rotation: rotating $\overrightarrow{AB}$ will lead to $\overrightarrow{BC}$, rotating $\overrightarrow{BC}$ will give $\overrightarrow{CA}$ etc...:

1.6 $$\frac{\overrightarrow{AB}}{|AB|}\,e^{\eta\beta} = \frac{\overrightarrow{BC}}{|BC|} \quad\text{and}\quad \frac{\overrightarrow{BC}}{|BC|}\,e^{\eta\gamma} = \frac{\overrightarrow{CA}}{|CA|} \quad\text{and}\quad \frac{\overrightarrow{CA}}{|CA|}\,e^{\eta\alpha} = \frac{\overrightarrow{AB}}{|AB|}$$

we can from these relation compute the surface using the geometrical algebra as a tool:

1.7 $$\begin{aligned} 4\,\eta\,S_\triangle &= [\overrightarrow{AB},\overrightarrow{BC}] = \frac{|BC|}{|AB|}\,[\overrightarrow{AB},\overrightarrow{AB}\;e^{\eta\beta}] \\ &= \frac{|BC|}{|AB|}\left(\overrightarrow{AB}\overrightarrow{AB}e^{\eta\beta} - \overrightarrow{AB}e^{\eta\beta}\overrightarrow{AB}\right) = |BC|\;|AB|\;\left(e^{\eta\beta} - e^{-\eta\beta}\right) = 2\,\eta\,|BC|\;|AB|\;\sin\beta \\ S_\triangle &= \frac{1}{2}\,|AB|\;|BC|\;\sin\beta = \frac{1}{2}\,|BC|\;|CA|\;\sin\gamma = \frac{1}{2}\,|CA|\;|AB|\;\sin\alpha \end{aligned}$$

It is also trivial to see another very classical result:

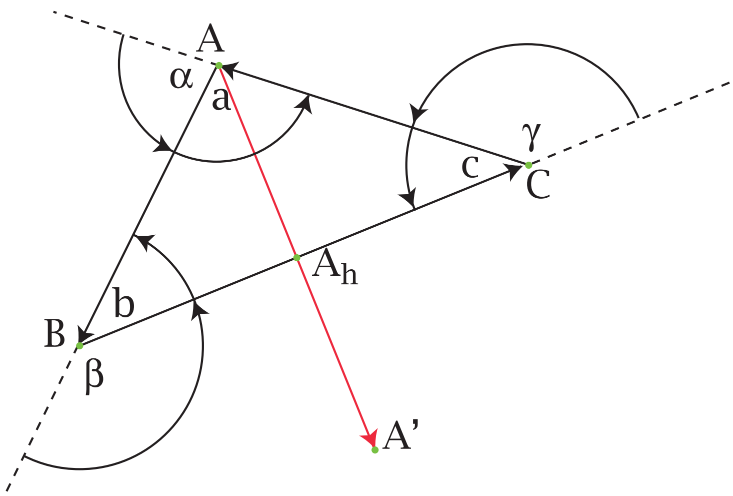
1.8 $$\{\overrightarrow{AB},\overrightarrow{BC}\} = \frac{|BC|}{|AB|}\,\{\overrightarrow{AB},\overrightarrow{AB}\;e^{\eta\beta}\} = \frac{|BC|}{|AB|}\left(\overrightarrow{AB}\overrightarrow{AB}e^{\eta\beta} + \overrightarrow{AB}e^{\eta\beta}\overrightarrow{AB}\right) = 2\;|AB|\;|BC|\;\cos\beta$$

If we introduce an extra point, say O, which could be the center of the coordinate system for example, we have the funny relation:

1.9 $$\begin{aligned} 4\,\eta\,S_{\triangle} = [\overrightarrow{AB},\overrightarrow{BC}] &= [\overrightarrow{AO}+\overrightarrow{OB},\overrightarrow{BO}+\overrightarrow{OC}] \\ &= [\overrightarrow{AO},\overrightarrow{BO}] + [\overrightarrow{AO},\overrightarrow{OC}] + [\overrightarrow{OB},\overrightarrow{OC}] \\ &= [\overrightarrow{OA},\overrightarrow{OB}] + [\overrightarrow{OB},\overrightarrow{OC}] + [\overrightarrow{OC},\overrightarrow{OA}] \\ &= [\overrightarrow{BO},\overrightarrow{OA}] + [\overrightarrow{CO},\overrightarrow{OB}] + [\overrightarrow{AO},\overrightarrow{OC}] \\ S_{ABC} &= S_{BOA} + S_{COB} + S_{AOC} \end{aligned}$$

2 The triangle iteration

Let's now build the iteration process. If ABC is a triangle. Let A' the symetric of A by respect to $\overrightarrow{BC}$. The vector $\overrightarrow{AA'}$ is $\perp$ to $\overrightarrow{BC}$ so we can directly write this property as: $\overrightarrow{AA'} = \lambda\,\eta\,\overrightarrow{BC}$ where λ is a real number.



The parameter λ is determine using the algebra properties and also $\overrightarrow{AA_h} = 2\;\overrightarrow{AA'}$, the surface of the triangle, and the fact that $\overrightarrow{BA_h}$ is $\parallel$ to $\overrightarrow{BC}$:

2.1 $$\begin{aligned} [\overrightarrow{AB},\overrightarrow{BC}] &= [\overrightarrow{AA_h}+\overrightarrow{A_hB};\overrightarrow{BC}] = [\overrightarrow{AA_h};\overrightarrow{BC}] = 4\,\eta\,S_{\triangle} \\ [\overrightarrow{AA'},\overrightarrow{BC}] &= 2\;[\overrightarrow{AB},\overrightarrow{BC}] \\ \lambda[\eta\,\overrightarrow{BC},\overrightarrow{BC}] &= 8\,\eta\,S_{\triangle} \\ \lambda\,\left(\eta\,\overrightarrow{BC}\,\overrightarrow{BC} - \overrightarrow{BC}\,\eta\,\overrightarrow{BC}\right) &= 8\,\eta\,S_{\triangle} \\ 2\,\lambda\,\eta\,|BC|^2 &= 8\,\eta\,S_{\triangle} \\ \lambda &= \frac{4\,S_{\triangle}}{|BC|^2} \end{aligned}$$

So the iteration process could be written as

$$2.2 \qquad \begin{cases} \overrightarrow{AA'} = \dfrac{4\,S_\triangle}{|BC|^2}\,\eta\,\overrightarrow{BC} \\ \overrightarrow{BB'} = \dfrac{4\,S_\triangle}{|CA|^2}\,\eta\,\overrightarrow{CA} \\ \overrightarrow{CC'} = \dfrac{4\,S_\triangle}{|AB|^2}\,\eta\,\overrightarrow{AB} \end{cases} \quad \rightarrow \quad \begin{cases} \overrightarrow{OA'} = \overrightarrow{OA} + \dfrac{4\,S_\triangle}{|BC|^2}\,\eta\,\overrightarrow{BC} \\ \overrightarrow{OB'} = \overrightarrow{OB} + \dfrac{4\,S_\triangle}{|CA|^2}\,\eta\,\overrightarrow{CA} \\ \overrightarrow{OC'} = \overrightarrow{OC} + \dfrac{4\,S_\triangle}{|AB|^2}\,\eta\,\overrightarrow{AB} \end{cases}$$

This is a very short expression of the iteration. We can even go one step futher and express:

$$2.3 \qquad \overrightarrow{A'B'} = \overrightarrow{AB} + \frac{4\,S_\triangle}{|CA|^2}\eta\overrightarrow{CA} - \frac{4\,S_\triangle}{|BC|^2}\eta\overrightarrow{BC}$$

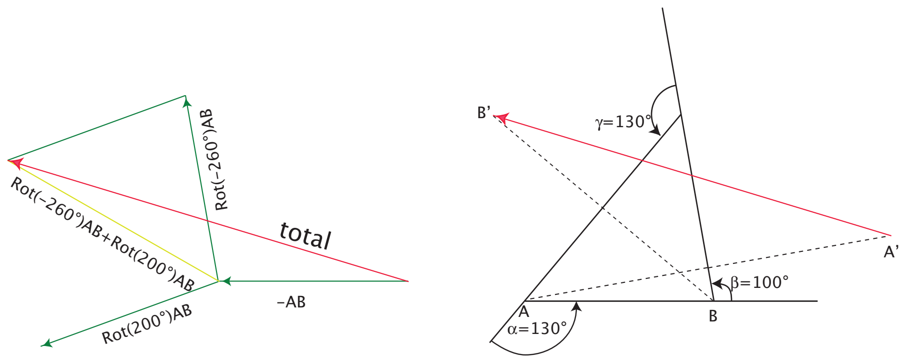
But

$$2.4 \qquad \overrightarrow{CA} = \frac{|CA|}{|AB|}\,\overrightarrow{AB}\,e^{-\eta\alpha} \quad \text{and} \quad \overrightarrow{BC} = \frac{|BC|}{|AB|}\,\overrightarrow{AB}\,e^{\eta\beta}$$

so

$$2.5 \qquad \begin{aligned} \overrightarrow{A'B'} &= \overrightarrow{AB} + \frac{4\,S_\triangle}{|CA|^2}\eta\frac{|CA|}{|AB|}\,\overrightarrow{AB}\,e^{-\eta\alpha} - \frac{4\,S_\triangle}{|BC|^2}\eta\frac{|BC|}{|AB|}\,\overrightarrow{AB}\,e^{\eta\beta} \\ &= \left(1 + \frac{4S_\triangle}{|CA||AB|}\eta\,e^{\eta\alpha} - \frac{4S_\triangle}{|BC||AB|}\eta\,e^{-\eta\beta}\right)\overrightarrow{AB} \\ &= \left(1 + 2\sin\alpha\,\eta\,e^{\eta\alpha} - 2\sin\beta\,\eta\,e^{-\eta\beta}\right)\overrightarrow{AB} \\ \overrightarrow{A'B'} &= \left(-1 + e^{2\eta\alpha} + e^{-2\eta\beta}\right)\,\overrightarrow{AB} \end{aligned}$$

which shows that the transformation is just adding to $-\overrightarrow{AB}$ the two rotated ⤹ vector by -2α and 2β (see below)



$$2.6 \qquad \overrightarrow{A'B'} = \left((-1 + \cos 2\alpha + \cos 2\beta) + \eta\ (\sin 2\alpha - \sin 2\beta)\right)\ \overrightarrow{AB}$$

this last expression is another way to express the relation between the two triangle and could be written as:

$$2.7 \qquad \overrightarrow{A'B'} = Z_{\alpha\beta}\,e^{\eta\,\xi_{\alpha\beta}}\,\overrightarrow{AB} \quad \text{and} \quad \overrightarrow{B'C'} = Z_{\beta\gamma}\,e^{\eta\,\xi_{\beta\gamma}}\,\overrightarrow{BC} \quad \text{and} \quad \overrightarrow{C'A'} = Z_{\gamma\alpha}\,e^{\eta\,\xi_{\gamma\alpha}}\,\overrightarrow{CA}$$

where Z correspond to a “dilatation” while $e^{\eta\,\xi}$ a rotation in the ⤸.

$$2.8 \qquad Z^2 = 3 - 2\cos[2\alpha] - 2\cos[2\beta] + 2\cos[2(\alpha+\beta)]$$

and of course:

2.9 $$Z\ \cos\xi = -1 + \cos 2\alpha + \cos 2\beta \quad \text{and} \quad Z\ \sin\xi = \sin 2\alpha - \sin 2\beta$$

The new length is given by

2.10 $$|A'B'|^2 = Z^2\ |AB|^2$$

An other way to compute the new length is provided by the first formula for the iteration [2.3].

2.11 $$|A'B'|^2 = \frac{1}{2}\{\overrightarrow{A'B'}, \overrightarrow{A'B'}\} = \frac{1}{2}\{\overrightarrow{AB} + \frac{4\ S_\triangle}{|CA|^2}\eta\overrightarrow{CA} - \frac{4\ S_\triangle}{|BC|^2}\eta\overrightarrow{BC}, \overrightarrow{AB} + \frac{4\ S_\triangle}{|CA|^2}\eta\overrightarrow{CA} - \frac{4\ S_\triangle}{|BC|^2}\eta\overrightarrow{BC}\}$$

after the expansion we find:

2.12 $$|A'B'|^2 = |AB|^2 - 8\ S_\triangle\ \sin 2\gamma$$

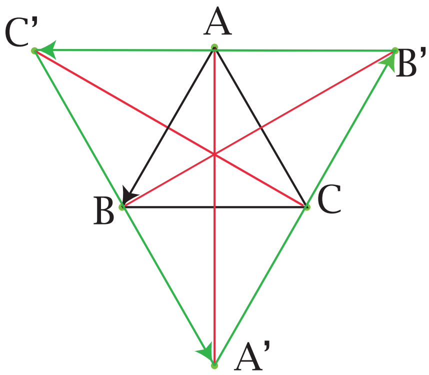
We can show easily that $Z_{max} = 3$ for the "zero" triangle $\alpha = \pi/2$, $\beta = \pi/2$ nd $\gamma = \pi$. So the maximum dilatation produced by the iteration on one side is 3. For the other side of the triangle we can compute

2.13 $$Z(\pi/2, \pi/2) = 3 \quad Z(\pi/2, \pi) = 1$$

For equilateral triangle we have for the 3 side the same relation:

2.14 $$Z = 2 \quad \text{and} \quad Z\ \cos\xi = -2 \quad Z\ \sin\xi = 0$$

so we have just a dilatation by a factor of 2 and a rotation of π of all the side of the triangle which remains then equilateral.



3 Surface evolution

Let's see what happens with the surface after the transformation $ABC \to A'B'C'$. We can do this, for example, by calculating

3.1 $$4\ \eta\ S'_\triangle = [\overrightarrow{A'B'}, \overrightarrow{B'C'}] = [\overrightarrow{OA'}, \overrightarrow{OB'}] + [\overrightarrow{OB'}, \overrightarrow{OC'}] + [\overrightarrow{OC'}, \overrightarrow{OA'}]$$

3.2
$$
\begin{aligned}
[\overrightarrow{A'B'},\overrightarrow{B'C'}] &= [\overrightarrow{AB}+\frac{4\,S_\triangle}{|CA|^2}\,\eta\,\overrightarrow{CA}-\frac{4\,S_\triangle}{|BC|^2}\,\eta\,\overrightarrow{BC},\overrightarrow{BC}+\frac{4\,S_\triangle}{|AB|^2}\,\eta\,\overrightarrow{AB}-\frac{4\,S_\triangle}{|CA|^2}\,\eta\,\overrightarrow{CA}]\\
&= [\overrightarrow{AB},\overrightarrow{BC}]+\frac{4\,S_\triangle}{|AB|^2}[\overrightarrow{AB},\eta\overrightarrow{AB}]-\frac{4\,S_\triangle}{|CA|^2}[\overrightarrow{AB},\eta\overrightarrow{CA}]+\frac{4\,S_\triangle}{|CA|^2}[\eta\overrightarrow{CA},\overrightarrow{BC}]+\frac{16S_\triangle^2}{|CA|^2|AB|^2}[\eta\overrightarrow{CA},\eta\overrightarrow{AB}]\\
&\quad-\frac{4S_\triangle}{|BC|^2}[\eta\overrightarrow{BC},\overrightarrow{BC}]-\frac{16S_\triangle^2}{|BC|^2|AB|^2}[\eta\overrightarrow{BC},\eta\overrightarrow{AB}]+\frac{16S_\triangle^2}{|BC|^2|CA|^2}[\eta\overrightarrow{BC},\eta\overrightarrow{CA}]\\
&= 4S_\triangle\eta-8S_\triangle\eta+\frac{4\,S_\triangle}{|CA|^2}\eta\{\overrightarrow{AB},\overrightarrow{CA}\}+\frac{4\,S_\triangle}{|CA|^2}\eta\{\overrightarrow{CA},\overrightarrow{BC}\}+\frac{16S_\triangle^2}{|CA|^2|AB|^2}[\overrightarrow{CA},\overrightarrow{AB}]\\
&\quad-8S_\triangle\eta-\frac{16S_\triangle^2}{|BC|^2|AB|^2}[\overrightarrow{BC},\overrightarrow{AB}]+\frac{16S_\triangle^2}{|BC|^2|CA|^2}[\overrightarrow{BC},\overrightarrow{CA}]
\end{aligned}
$$

We have

3.3
$$
\begin{aligned}
&[\overrightarrow{CA};\overrightarrow{AB}] = [\overrightarrow{CB}+\overrightarrow{BA},\overrightarrow{AB}] = [\overrightarrow{AB},\overrightarrow{BC}] = 4\,\eta\,S_\triangle\\
&[\overrightarrow{BC},\overrightarrow{AB}] = -4\,\eta\,S_\triangle\\
&[\overrightarrow{BC},\overrightarrow{CA}] = 4\,\eta\,S_\triangle\\
&\{\overrightarrow{AB},\overrightarrow{CA}\}+\{\overrightarrow{CA},\overrightarrow{BC}\} = \{\overrightarrow{AC},\overrightarrow{CA}\} = -2\,CA^2
\end{aligned}
$$

So finally

3.4
$$
\begin{aligned}
S'_\triangle &= -5\,S_\triangle+4\,S_\triangle\left(\sin^2\alpha+\sin^2\beta+\sin^2\gamma\right)\\
S'_\triangle &= -5\,S_\triangle+16\,S_\triangle^3\left(\frac{1}{AB^2BC^2}+\frac{1}{BC^2CA^2}+\frac{1}{CA^2AB^2}\right)
\end{aligned}
$$

For equilateral traingle we have, as said before:

3.5
$$
S'_\triangle = -5\,S_\triangle+4\,S_\triangle\frac{9}{4} = 4\,S_\triangle
$$

so after k iteration $S_k = 4^k\,S_0$.