Monthly Problem 88

Let $f : \mathbb{R} \to \mathbb{R}$. Suppose that for all $x, y \in \mathbb{R}$, $f(x+y) + f(x-y) = 2f(x) + 2f(y)$. Find $f(\frac{22}{7})$.

Solution

Let $x \in \mathbb{R}$. We show that $f(rx) = r^2 f(x)$ for all $r \in \mathbb{Q}$.

1. $f(0+0) + f(0-0) = 2f(0) + 2f(0)$, so $f(0) = 0$. Thus, $f(0x) = 0^2 f(x)$ and $f(1x) = 1^2 f(x)$.
   Now, if $f(kx) = k^2 f(x)$ for $k \in \{n-1, n\}$, then $f((n+1)x) = 2f(nx) + 2f(x) - f((n-1)x) = (n+1)^2 f(x)$.
   By induction, $f(nx) = n^2 f(x)$ for all $n \in \mathbb{N}$. Hence, $f(x) = f(n\frac{x}{n}) = n^2 f(\frac{x}{n})$, so $f(\frac{x}{n}) = \frac{1}{n^2} f(x)$.
   It follows that $f(\frac{a}{b}) = (\frac{a}{b})^2 f(1)$ for all $a, b \in \mathbb{N}$.
2. $f(0+u) + f(0-u) = 2f(0) + 2f(u)$, so $f(-u) = f(u)$. Therefore $f(r) = r^2 f(1)$ for all $r \in \mathbb{Q}$.
3. If $f(1) = 1$, then $f(\frac{22}{7}) = (\frac{22}{7})^2$. ■

Comment

We claim that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y)+f(x-y) = 2f(x)+2f(y)$ if and only if $f(x) = \varphi(x,x)$, where $\varphi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is additive in each argument. Viewing φ as a generalized binary product, f can be viewed as a generalized square function.
Proof:

1. Suppose $f(x+y) + f(x-y) = 2f(x) + 2f(y)$. Define $\varphi(x,y) = \frac{1}{2}[f(x+y) - f(x) - f(y)]$. By the lemma below,

$$\begin{aligned}
\varphi(x+y,z) &= [f(x+y+z) - f(x+y) - f(z)]/2 \\
&= [f(x+y) + f(x+z) + f(y+z) - f(x) - f(y) - f(z) - f(x+y) - f(z)]/2 \\
&= [f(x+z) - f(x) - f(z)]/2 + [f(y+z) - f(y) - f(z)]/2 \\
&= \varphi(x,z) + \varphi(y,z)
\end{aligned}$$

   Thus, φ is additive in the first argument. Since $\varphi(x,y) = \varphi(y,x)$, it is additive in the second argument. Finally, $\varphi(x,x) = \frac{1}{2}[f(2x) - 2f(x)] = \frac{1}{2}[2^2 f(x) - 2f(x)] = f(x)$.

2. Suppose that $\varphi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is additive in each argument, and that $f(x) = \varphi(x,x)$. Then

$$\begin{aligned}
f(x+y) = \varphi(x+y, x+y) = \varphi(x,x) + \varphi(x,y) + \varphi(y,x) + \varphi(y,y) \\
f(x-y) = \varphi(x-y, x-y) = \varphi(x,x) - \varphi(x,y) - \varphi(y,x) + \varphi(y,y)
\end{aligned}$$

   Adding, we see that $f(x+y) + f(x-y) = 2\varphi(x,x) + 2\varphi(y,y) = 2f(x) + 2f(y)$. ■

The simplest bi-additive φ would be of the form $\varphi(x,y) = xy\varphi(1,1)$, which would correspond to $f(x) = x^2 f(1)$. However, we can construct much more complicated bi-additive φ as follows. By the maximum principle, there exists a basis $\mathbb{B}$ for $\mathbb{R}$ considered as a vector space over $\mathbb{Q}$. Define φ arbitrarily on $\mathbb{B}\times\mathbb{B}$. Now, any $x, y \in \mathbb{R}$ can be uniquely written $x = \sum_{i=1}^m \xi_i b_i$ and $y = \sum_{j=1}^n \gamma_j b_j$, where $\xi_i, \gamma_j \in \mathbb{Q}$ and $b_i, b_j \in \mathbb{B}$. Define $\varphi(x,y) = \sum_{i=1}^m \sum_{j=1}^n \xi_i \gamma_j \varphi(b_i, b_j)$. By construction, φ is additive in each argument. Since φ can be defined arbitrarily on $\mathbb{B}\times\mathbb{B}$, the corresponding f can be quite pathological.

Lemma. Suppose $f(x+y)+f(x-y) = 2f(x)+2f(y)$. Then $f(x+y+z) = f(x+y)+f(x+z)+f(y+z)-f(x)-f(y)-f(z)$.
Proof. Adding the following equations,

$$\begin{aligned}
f(x+y+z) + f(x+y-z) &= 2f(x+y) + 2f(z) \\
f(x+y+z) + f(x-y+z) &= 2f(x+z) + 2f(y) \\
f(x+y+z) + f(y+z-x) &= 2f(y+z) + 2f(x) \\
f(x) + f(y-z) &= [f(x+(y-z)) + f(x-(y-z))]/2 \\
f(z) + f(x-y) &= [f(z+(x-y)) + f(z-(x-y))]/2 \\
f(y) + f(x-z) &= [f(y+(x-z)) + f(y-(x-z))]/2 \\
2f(y) + 2f(z) &= f(y+z) + f(y-z) \\
2f(x) + 2f(y) &= f(x+y) + f(x-y) \\
2f(x) + 2f(z) &= f(x+z) + f(x-z)
\end{aligned}$$

we obtain $3f(x+y+z) + 3f(x) + 3f(y) + 3f(z) = 3f(x+y) + 3f(x+z) + 3f(y+z)$, and the lemma follows. ■